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Scaling and *s*-Channel Helicity Conservation via Optimal State Description of Hadron–Hadron Scattering

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Two important physical laws of hadron-hadron scattering—the scaling of the angular distributions and *s*-channel helicity conservation—are proved using reproducing-kernel Hilbert space methods. All the results are obtained as special properties of optimal state dominance in hadron-hadron scattering.

1. INTRODUCTION

The present paper may be considered as a continuation and an extension of our previous results (Ion and Scutaru, 1985; Ion, 1985) in which the two-body scattering amplitude is assumed to be an element of a functional Hilbert space called *reproducing kernel Hilbert space* (RKHS) (see, e.g., Aronszajn, 1943, 1950; Bergman and Schiffer, 1953; Krein, 1940, 1949, 1963; Meschkowski, 1962; Parzen, 1967; Shapiro, 1971; Hille, 1972; Higgins, 1972, 1977). We have shown that: (i) the RKHS has many special properties that make it an adequate variational space for the description of the scattering in terms of an optimum principle; (ii) the notion of optimal scattering state (Ion, 1982a, b) and the reproducing kernel (RK) of the RKHS associated to the scattering amplitude are the same; (iii) the expansion of the scattering amplitude in terms of optimal states is an important alternative to partial wave analysis; (iv) the essential characteristic features of the scattering as predicted by optimal state dominance are satisfied experimentally to surprising accuracy (see Ion, 1982a,b, 1985) for all pp, $\bar{p}p$, $\pi^{\pm}p$, $K^{\pm}p$ scattering at all energies higher than 2 GeV; (v) the dual diffractive scattering (DDS)

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and dual diffractive resonance (DDR) phenomena (Ion, 1981a,b; Ion and Ion-Mihai 1981a,b) are described in a unified manner using RKHS methods.

On the other hand, it is interesting to note that RKHS methods were also used effectively in, e.g., the theory of coherent states (Bargmann, 1961; McKenna and Klauder, 1964; Klauder and Sadarshan, 1968; Perelomov, 1972; Scutaru, 1977; Klauder and Skagerstam, 1985), group representations (e.g., Carey, 1977, 1978), stochastic quantum mechanics (e.g., Prugovecki, 1983; Ali, 1984a,b; Schroeck, 1984), elementary particle physics (Cutkosky, 1973; Okubo, 1974), and signal processing (e.g., Weinert, 1983). It was recognized (McKenna and Klauder, 1964; Klauder and Sudarshan, 1968; Perelomov, 1972) that the notion of the coherent state (Glauber, 1963a,b; McKenna and Klauder, 1964) and the reproducing kernel of the Hilbert space of wave functions are the same. In this respect the optimal state from the Hilbert space of helicity amplitudes is analogous to the coherent state from the Hilbert space of wave functions.

In this paper two important physical laws of hadron-hadron scattering—the scaling law of the angular distribution and the s-channel helicity conservation—are derived in terms of optimal states by using RKHS methods. In Section 2, we discuss briefly some essential definitions and results on the RKHS and give the extremal and scaling properties for the scattering of spinless particles. In Section 3, these properties as well as s-channel helicity conservation are established for the scattering of particles with arbitrary spins; conclusions are summarized in Section 4.

2. SCALING PROPERTY OF OPTIMAL STATES

Let $f(x)$, $x \in [-1, +1]$, be the scattering amplitude of a two-body scattering process

$$a + b \to a + b \tag{1}$$

where a and b are spinless hadrons and $x \equiv \cos\theta$; θ is the center-of-mass (CM) scattering angle. The normalization of $f(x)$ is chosen such that the differential cross section $(d\sigma/d\Omega)(x)$, the elastic integrated cross section σ_{el}, and the total cross section σ_T are given by

$$\frac{d\sigma}{d\Omega}(x) = |f(x)|^2, \qquad x \in [-1, +1] \tag{2}$$

$$\sigma_{el} = 2\pi \int_{-1}^{+1} \frac{d\sigma}{d\Omega}(x)\, dx = 2\pi \int_{-1}^{+1} |f(x)|^2\, dx \tag{3}$$

$$\sigma_T = 4\pi\lambda\!\!\!^{-}\, \mathrm{Im}\, f(1) \tag{4}$$

where $\lambda\!\!\!^{-} = 1/p$, with p the CM momentum. The energy dependence of $f(x)$, $(d\sigma/d\Omega)(x)$, σ_{el}, and σ_T was suppressed since we work at a fixed value of this variable.

Let H be the Hilbert space of the scattering amplitude defined on the interval $S \equiv [-1, +1]$ with the inner product $\langle \cdot, \cdot \rangle$, and the norm $\|\cdot\|$ given by

$$\langle f, g \rangle = \int_{-1}^{+1} f(x)\bar{g}(x)\, dx, \qquad \forall f, g \in H \tag{5a}$$

$$\langle f, f \rangle = \int_{-1}^{+1} |f(x)|^2\, dx = \|f\|^2 < \infty \tag{5b}$$

Now, we review briefly some of the definitions and results on the RKHS that we shall use in this investigation.

Definition 1. A Hilbert space H of complex-valued functions defined on a set S is said to be RKHS if it enjoys the following reproducing property. There exists a complex-valued function $K(x, y)$ on $S \times S$, called the RK, such that

(i) for any fixed $y \in S$, K_y is in H (6a)

(ii) $K_y(x) = K(x, y)$ induces the reproducing property

$$\langle f, K_y \rangle = f(y) \tag{6b}$$

for each $f \in H$, and $y \in S$.

K_y is called the reproducing element for the point y, while the totality of elements K_y is the RK of H.

Definition 2. Let H be the RKHS of the scattering amplitude f of the process (1) and let K be the RK of H. The scattering state of the system (1) described by the amplitude

$$f_y = f(y)\frac{K_y}{K(y, y)}, \qquad K(y, y) \neq 0, \quad f(y) \neq 0, \quad y \in S \tag{7}$$

is called the optimal state for the point y.

Corollary 1. (Ion and Scutaru, 1985). If the scattering amplitude f is an element of the RKHS H with the RK K, then the functionals (2) and (3) must obey the inequality

$$\frac{d\sigma}{d\Omega}(y) \leq \frac{\sigma_{\text{el}}}{2\pi} K(y, y) \tag{8}$$

the equality holding in (8) if and only if the scattering amplitude is the optimal state (7).

We note that most of the important properties of the RKHS were discussed recently by Ion (1985) in the context of the optimal state description of hadron–hadron scattering. We saw that a number of essential characteristic features which are common to all optimal states (7) are direct consequences of the RK properties, while for any specific example of optimal states a corresponding set of additional properties holds true.

Here we discuss one of the most important characteristic features of the scattering amplitude, *the scaling property*, which can be derived via optimal state dominance.

Theorem 1. Let $f(x)$ be the scattering amplitude of the process (1) written in terms of the partial ampltude f_l as

$$f(x)=\sum_{l=0}^{L}(2+1)f_lP_l(x), \qquad x\in[-1,+1], \quad f_l\in C \tag{9}$$

where $P_l(x)$, $l=0,1,\ldots,L$, are Legendre polynomials. Then: (i) the scattering amplitude is an element of the RKHS H defined on the interval $S\equiv[-1,+1]$ if and only if L is finite. (ii) H possesses the reproducing kernel

$$\begin{aligned}K(x,y)&=\sum_{l=0}^{L}(l+\tfrac{1}{2})P_l(x)P_l(y)\\&=\tfrac{1}{2}(L+1)\frac{P_{L+1}(x)P_L(y)-P_L(x)P_{L+1}(y)}{x-y}\end{aligned} \tag{10a}$$

$$K(y,y)=\tfrac{1}{2}(L+1)[\dot{P}_{L+1}(y)P_L(y)-\dot{P}_L(y)P_{L+1}(y)] \tag{10b}$$

where $\dot{P}_L(x)\equiv dP_l(x)/dx$.

Proof. The first part of the theorem is obtained by observing that the evaluation functional $f(y)$ is bounded on H if L is finite. Indeed,

$$|f(y)|\leq\|f\|\left[\sum_{l=0}^{L}(l+1/2)P_l^2(y)\right]^{1/2}\leq\|f\|(L+1)/\sqrt{2}$$

So, by Theorem 1 (Ion and Scutaru, 1985), H is an RKHS with $K(x,y)$ given by (10a), (10b). One can verify that $K_y\in H$ and also the reproducing property

$$\begin{aligned}\langle f,K_y\rangle&=\int_{-1}^{+1}f(x)\,\overline{K_y(x)}\,dx\\&=\frac{1}{2}\sum_{l=0}^{L}\sum_{m=0}^{L}(2l+1)(2m+1)f_lP_m(y)\int_{-1}^{+1}P_l(x)P_m(x)\,dx\end{aligned}$$

$$= \sum_{l=0}^{L} (2l+1) f_l P_l(y) = f(y) \quad \blacksquare$$

Some of the main results obtained in this section can be summarized as follows:

Theorem 2. Assume that the scattering amplitude f is an element of the RKHS H that possesses the RK (10a), (10b).

(i) If σ_{el} and $(d\sigma/d\Omega)(y)$, $y \in [-1, +1]$, are known from the experimental data, then any cut off L on the total angular momentum must obey the bound

$$\frac{4\pi}{\sigma_{el}} \frac{d\sigma}{\delta\Omega}(y) \leq (L+1)[\dot{P}_{L+1}(y) P_L(y) - \dot{P}_L(y) P_{L+1}(y)] \tag{11a}$$

(ii) The equality holds in (11a) if and only if f is the optimal scattering amplitude (7), i.e.,

$$f(x) = f(y) \frac{K(x, y)}{K(y, y)} = f(y) \frac{P_{L+1}(x) P_L(y) - P_L(x) P_{L+1}(y)}{(x-y)[\dot{P}_{L+1}(y) P_L(y) - \dot{P}_L(y) P_{L+1}(y)]} \tag{11b}$$

where L is the solution of the equation

$$\frac{4\pi}{\sigma_{el}} \frac{d\sigma}{d\Omega}(y) = 2K(y, y) = (L+1)[\dot{P}_{L+1}(y) P_L(y) - \dot{P}_L(y) P_{L+1}(y)] \tag{11c}$$

(iii) The logarithmic slope of the angular distribution of the optimal state (11b) at a point $t_z \equiv -2p^2(1-z)$, $z \in [-1, +1]$, is given by

$$\begin{aligned} b_{t_z} &\equiv \frac{d}{dt}\left[\ln \frac{d\sigma}{d\Omega}(s, t)\right]\Big|_{t=t_z} \\ &= \lambda\!\!\!^{-2}\{[\dot{P}_{L+1}(z) P_L(y) - \dot{P}_L(z) P_{L+1}(y)](z-y) \\ &\quad - P_{L+1}(z) P_L(y) + P_L(z) P_{L+1}(y)\} \\ &\quad \times \{(z-y)[P_{L+1}(z) P_L(y) - P_L(z) P_{L+1}(y)]\}^{-1} \end{aligned} \tag{11d}$$

Proof. The results (11a)-(11d) are obtained from Corollary 1, using equations (7), (8), and (10a), (10b). ■

Corollary 2. Assume that the scattering amplitude f is an element of the RKHS H that possesses the RK (10a), (10b).

(i) If σ_{el} and σ_T are given, then any cutoff L on the total angular momentum must obey the bound

$$(L+1)^2 \geq \sigma_T^2 / 4\pi \lambda\!\!\!^{-2} \sigma_{el} \tag{12a}$$

(ii) The equality holds in (12) if and only if f is the optimal amplitude

$$f(x)=i\frac{\sigma_T}{4\pi\lambda}\frac{K(x,1)}{K(1,1)}$$
$$=i\frac{\sigma_T}{4\pi\lambda}\frac{1}{L+1}\frac{P_{L+1}(x)-P_L(x)}{x-1}$$
$$=i\frac{\sigma_T}{4\pi\lambda}\frac{1}{(L+1)^2}[\dot{P}_{L+1}(x)+\dot{P}_L(x)] \qquad (12b)$$

where

$$L=\text{integer}\{(\sigma_T/4\pi\lambda^2\sigma_{el})^2-1\} \qquad (12c)$$

(iii) The forward diffraction peak of the optimal amplitude (12b), (12c) possesses the scaling property

$$\frac{1}{(d\sigma/d\Omega)(1)}\frac{d\sigma}{d\Omega}(x)=\left[\frac{J_1(\tau_t)}{\tau_t}\right]^2 \quad \text{for small} \quad \tau_t \qquad (13a)$$

where

$$\frac{d\sigma}{d\Omega}(1)=\left(\frac{\sigma_T}{4\pi\lambda}\right)^2 \qquad (13b)$$

$$\tau_t=2(|t|b_t)^{1/2}=\left[|t|\left(\frac{\sigma_T^2}{4\pi\sigma_{el}}-\lambda^2\right)\right]^{1/2} \qquad (13c)$$

and where b_t is the logarithmic slope of the forward diffraction peak and $J_1(\tau_t)$ is the Bessel function of first order.

(iv) The backward diffraction peak of the optimal state (12b), (12c) possesses the scaling property

$$\frac{1}{(d\sigma/d\Omega)(-1)}\frac{d\sigma}{d\Omega}(x)=\left[\frac{J_1(\tau_u)}{\tau_u}\right]^2 \quad \text{for small } \tau_u \qquad (14a)$$

where

$$\frac{d\sigma}{\delta\Omega}(-1)=\frac{\sigma_{el}}{4\pi} \qquad (14b)$$

$$\tau_u=2(|u|b_u)^{1/2}=\left[2|u|\left(\frac{\sigma_T^2}{4\pi\sigma_{el}}-\lambda^2\right)\right]^{1/2} \qquad (14c)$$

and where $u=-2p^2(1+x)$ is the usual u-transfer momentum, and b_u is the logarithmic slope of the backward diffraction peak.

Proof. The results (12a)–(12c) are derived from (11a)–(11c) for $y=1$ and the usual inequality (Wick, 1943)

$$\left(\frac{\sigma_T}{4\pi\lambda\!\!\!^{-}}\right)^2 \leq \frac{d\sigma}{d\Omega}(1)$$

Then, equations (13a)–(13c) and (14a)–(14c) are obtained in a direct way using the properties of Legendre polynomials and the relation

$$P_l(x) \approx J_0[(2l+1)\sin\tfrac{1}{2}\theta] \qquad \text{for } l \gg 1 \text{ and small angles} \quad \blacksquare$$

Corollary 3. Assume that the scattering amplitude f of the process (1) is an element of the RKHS H that possesses the RK (10a), (10b).

(i) If σ_{el} and $(d\sigma/d\Omega)(-1)$ are given, then any cutoff L on the total angular momentum must obey the inequality

$$(L+1)^2 \geq \frac{4\pi}{\sigma_{el}} \frac{d\sigma}{d\Omega}(-1) \tag{15a}$$

(ii) The equality holds in (15a) if and only if the scattering amplitude f is given by

$$f(x) = f(-1)\frac{K(x,-1)}{K(-1,-1)} = f(-1)\frac{(-1)^L}{L+1}\frac{P_{L+1}(x)+P_L(x)}{x+1} \tag{15b}$$

where

$$L = \text{integer}\left\{\left[\frac{4\pi}{\sigma_{el}}\frac{d\sigma}{d\Omega}(-1)\right]^{1/2} - 1\right\} \tag{15c}$$

(iii) The forward diffraction peak of the optimal state (15b), (15c) possesses the scaling property

$$\frac{1}{(d\sigma/d\Omega)(1)}\frac{d\sigma}{d\Omega}(x) = \left[\frac{J_1(\tau_t)}{\tau_t}\right]^2 \qquad \text{for small} \quad \tau_t \tag{16a}$$

where

$$\frac{d\sigma}{d\Omega}(1) = \frac{\sigma_{el}}{4\pi} \tag{16b}$$

and

$$\tau_t = 2(|t|b_t)^{1/2} = \left\{2\lambda\!\!\!^{-2}|t|\left[\frac{4\pi}{\sigma_{el}}\frac{d\sigma}{d\Omega}(-1) - 1\right]\right\}^{1/2} \tag{16c}$$

(iv) The backward diffraction peak of the optimal state (15b), (15c) possesses the scaling property

$$\frac{1}{(d\sigma/d\Omega)(-1)}\frac{d\sigma}{d\Omega}(x)=\left(\frac{J_1(\tau_u)}{\tau_u}\right)^2 \qquad \text{for small} \quad \tau_u \tag{17a}$$

and

$$\tau_u=2(|u|b_u)^{1/2}=\left\{\lambda\!\!\!^{-2}|t|\left[\frac{4\pi}{\sigma_{\text{el}}}\frac{d\sigma}{d\Omega}(-1)-1\right]\right\}^{1/2} \tag{17b}$$

Proof. The results (15a)-(15c) are obtained from (11a), (11b) since $K(-1,-1)=(L+1)^2/2$, while the optimal predictions (16a)-(16c) and (17a), (17b) are derived in a straightforward way using the properties of the Legendre polynomials. ■

3. SCALING AND *s*-CHANNEL HELICITY CONSERVATION

Now let us consider the process (1) in which the particles a and b have the spins s_a and s_b, respectively. Then, for the description of the system (1) we use the helicity formalism of Jacob and Wick (1959). Let

$$f^{[\mu]}(x)=\langle\mu'_a\mu'_b|F(s,t)|\mu_a\mu_b\rangle, \qquad [\mu]\equiv(\mu_a\mu_b;\mu'_a\mu'_b) \tag{18}$$

be the helicity amplitude of the process (1) with the initial helicities μ_a and μ_b and the final helicities μ'_a, μ'_b, where s and t are the squares of the CM energy and transfer momentum variables. The normalization is chosen such that for each helicity channel $[\mu]\equiv(\mu_a\mu_b;\mu'_a\mu'_b)$ we have

$$\frac{d\sigma^{[\mu]}}{d\Omega}(x)=|f^{[\mu]}(x)|^2, \qquad x\in[-1,+1] \tag{19a}$$

$$\sigma_{\text{el}}^{[\mu]}=2\pi\int_{-1}^{+1}|f^{[\mu]}|^2\,dx=2\pi\|f^{[\mu]}\|^2 \tag{19b}$$

while the unpolarized (differential, elastic, and integrated) cross sections are given by

$$\frac{d\sigma}{d\Omega}(x)=\frac{1}{(2s_a+1)(2s_b+1)}\sum_{[\mu]}\frac{d\sigma^{[\mu]}}{d\Omega}(x), \qquad x\in[-1,+1] \tag{20a}$$

$$\sigma_{\text{el}}=\frac{1}{(2s_a+1)(2s_b+1)}\sum_{[\mu]}\sigma_{\text{el}}^{[\mu]} \tag{20b}$$

$$\sigma_T=\frac{1}{(2s_a+1)(2s_b+1)}\sum_{[\mu_0]}\sigma_T^{[\mu_0]} \tag{20c}$$

where $[\mu_0]\equiv(\mu_a\mu_b;\mu_a\mu_b)$ are the helicity-conserving channels.

We recall that, for each helicity-conserving channel $[\mu_0]$, $\operatorname{Im} f^{[\mu_0]}(1)$ and $\sigma_T^{[\mu_0]}$ are related via the optical theorem

$$\sigma_T^{[\mu_0]} = 4\pi\lambda\!\!\!{}^{-} \operatorname{Im} f^{[\mu_0]}(1) \tag{21}$$

Next, let us consider that each helicity amplitude $f^{[\mu]}$ is an element of the RKHS $H^{[\mu]}$ defined on the interval $[-1, +1]$ with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$ given by

$$\langle f^{[\mu]}, g^{[\mu]} \rangle = \int_{-1}^{+1} f^{[\mu]}(x) \overline{g^{[\mu]}}(x)\, dx, \qquad f^{[\mu]}, g^{[\mu]} \in H^{[\mu]} \tag{22a}$$

and

$$\langle f^{[\mu]}, f^{[\mu]} \rangle = \int_{-1}^{+1} |f^{[\mu]}(x)|^2\, dx = \| f^{[\mu]} \|^2 < \infty \tag{22b}$$

Definition 3. Let $H^{[\mu]}$ be the RKHS of the helicity amplitude $f^{[\mu]}$ of the process (1) and let $K^{[\mu]}$ be the RK of $H^{[\mu]}$. The scattering state described by the helicity amplitude

$$f_y^{[\mu]} = f^{[\mu]}(y) \frac{K_y^{[\mu]}}{K^{[\mu]}(y, y)}, \qquad K^{[\mu]}(y, y) \neq 0, \qquad f^{[\mu]}(y) \neq 0 \tag{23}$$

is called the *optimal state of the channel* $[\mu]$ for the point y.

The following result describes the extremal property of the optimal state (23).

Corollary 4. If each helicity amplitude $f^{[\mu]}$ is an element of the RKHS $H^{[\mu]}$ with RK $K^{[\mu]}$, then the functionals (19a), (19b), (20a), (20b) must obey the bounds

$$\frac{d\sigma^{[\mu]}}{d\Omega}(y) \leq \frac{\sigma_{\text{el}}^{[\mu]}}{2\pi} K^{[\mu]}(y, y), \qquad y \in [-1, +1] \tag{24a}$$

and

$$\frac{d\sigma}{d\Omega}(y) \leq \frac{\sigma_{\text{el}}}{2\pi} K^{[\mu]}(y, y), \qquad y \in [-1, +1] \tag{24b}$$

for all $[\mu]$ for which $K^{[\mu]}(y, y) \neq 0$, respectively. The equality holds in (24a), (24b) if and only if the helicity amplitude $f^{[\mu]}$ is the optimal state (23) for each channel $[\mu]$ in the case (24a) and for all channels $[\mu]$ in the case (24b).

Theorem 3. Let $f^{[\mu]}$ be the helicity amplitude of the process (1) described in terms of the partial amplitudes $f_j^{[\mu]}$ by

$$f^{[\mu]}(x) = \sum_{J_{\min}}^{J} (2j+1) f_j^{[\mu]} d_{\mu\nu}^j(x) \tag{25a}$$

where $\{d_{\mu\nu}^j(x), x \in [-1, +1]\}$ is the set of rotation functions (see Rose, 1957;

Edmonds, 1957) and

$$J_{\min} \equiv \max\{|\mu|, |\nu|\}; \qquad \mu = \mu_a - \mu_b, \quad \nu = \mu'_a - \mu'_b \tag{25b}$$

Then: (i) $f^{[\mu]}$ is an element of RKHS $H^{[\mu]}$ defined on $[-1, +1]$ if and only if J is finite, and (ii) $H^{[\mu]}$ possesses the reproducing kernel

$$\begin{aligned} K^{[\mu]}(x, y) &= \sum_{J_{\min}}^{J} (j+\tfrac{1}{2})\, d^j_{\mu\nu}(x)\, d^j_{\mu\nu}(y) \\ &= \frac{[(J+1)^2-\mu^2]^{1/2}[(J+1)^2-\nu^2]^{1/2}}{2(J+1)} \\ &\quad \times \frac{d^{J+1}_{\mu\nu}(x)\, d^J_{\mu\nu}(y) - d^J_{\mu\nu}(x)\, d^{J+1}_{\mu\nu}(y)}{x-y} \end{aligned} \tag{26a}$$

$$\begin{aligned} K^{[\mu]}(y, y) &= \sum_{J_{\min}}^{J} (j+\tfrac{1}{2})[d^j_{\mu\nu}(y)]^2 \\ &= \frac{[(J+1)^2-\mu^2]^{1/2}[(J+1)^2-\nu^2]^{1/2}}{2(J+1)} [\dot{d}^{J+1}_{\mu\nu}(y)\, d^J_{\mu\nu}(y) \\ &\quad - \dot{d}^J_{\mu\nu}(y)\, d^{J+1}_{\mu\nu}(y)] \end{aligned} \tag{26b}$$

Proof. Indeed, from the reproducing property

$$\langle f^{[\mu]}, K^{[\mu]}_y \rangle = f^{[\mu]}(y)$$

using Schwarz' inequality, we get

$$\begin{aligned} |f^{[\mu]}(y)| &\leq \|f^{[\mu]}\| [K^{[\mu]}(y, y)]^{1/2} \\ &\leq \|f^{[\mu]}\| [(J+1)^2 - J^2_{\min}]^{1/2}/\sqrt{2} \end{aligned} \tag{27}$$

since

$$K^{[\mu]}(y, y) \leq K^{[\mu_0]}(1, 1) = \tfrac{1}{2}[(J+1)^2 - J^2_{\min}]$$

Hence, the evaluation functional $f^{[\mu]}(y)$ is bounded on $H^{[\mu]}$ if J is finite. Then, $H^{[\mu]}$ is an RKHS with the RK $K^{[\mu]}$ given by (26a), (26b), since the set $\{d^j_{\mu\nu}(y), y \in [-1, +1]\}$ is a complete orthonormal sequence in the RKHS $H^{[\mu]}$. ■

Theorem 4. Assume that the helicity amplitude $f^{[\mu]}$ of the process (1) is an element of the RKHS $H^{[\mu]}$ that possesses the RK $K^{[\mu]}$, (26a), (26b). Then:

(i) If $\sigma^{[\mu]}_{\text{el}}$ and $(d\sigma^{[\mu]}/d\Omega)(y)$, $y \in [-1, +1]$, are given, then any cutoff J_μ on the total angular momentum j must obey the bound

$$\begin{aligned} \frac{4\pi}{\sigma^{[\mu]}_{\text{el}}} \frac{d\sigma^{[\mu]}}{d\Omega}(y) &\leq \frac{[(J_\mu+1)^2-\mu^2]^{1/2}[(J_\mu+1)^2-\nu^2]^{1/2}}{2(J_\mu+1)} \\ &\quad \times [\dot{d}^{J_\mu+1}_{\mu\nu}(y)\, d^{J_\mu}_{\mu\nu}(y) - \dot{d}^{J_\mu}_{\mu\nu}(y)\, d^{J_\mu+1}_{\mu\nu}(y)] \end{aligned} \tag{28}$$

(ii) The equality holds in (28) if and only if $f^{[\mu]}$ is the optimal amplitude

$$f^{[\mu]}(x)=f^{[\mu]}(y)\frac{K^{[\mu]}(x,y)}{K^{[\mu]}(y,y)}$$
$$=f^{[\mu]}(y)\frac{d_{\mu\nu}^{J_\mu+1}(x)\,d_{\mu\nu}^{J_\mu}(y)-d_{\mu\nu}^{J_\mu}(x)\,d_{\mu\nu}^{J_\mu+1}(y)}{(x-y)[\dot{d}_{\mu\nu}^{J_\mu+1}(y)\,d_{\mu\nu}^{J_\mu}(y)-\dot{d}_{\mu\nu}^{J_\mu}(y)\,d_{\mu\nu}^{J}(y)]} \tag{29}$$

where J_μ is the solution of the equation

$$\frac{4\pi}{\sigma_{\text{el}}^{[\mu]}}\frac{d\sigma^{[\mu]}}{d\Omega}(y)=2K^{[\mu]}(y,y)$$
$$=\frac{[(J_\mu+1)^2-\mu^2]^{1/2}[(J_\mu+1)^2-\nu^2]^{1/2}}{J_\mu+1}$$
$$\times[\dot{d}_{\mu\nu}^{J_\mu+1}(y)\,d_{\mu\nu}^{J_\mu}(y)-\dot{d}_{\mu\nu}^{J_\mu}(y)\,d_{\mu\nu}^{J_\mu+1}(y)] \tag{30}$$

(iii) The logarithmic slope of the angular distribution of the optimal state (29) at a point $t_z\equiv-2p^2(1-z)$, $z\in[-1,+1]$, is given by

$$b_{t_z}^{[\mu]}\equiv\frac{d\sigma}{dt}\left[\ln\frac{d\sigma^{[\mu]}}{d\Omega}(x)\right]\Bigg|_{t=t_z}$$
$$=\lambda^2\{[\dot{d}_{\mu\nu}^{J_\mu+1}(z)\,d_{\mu\nu}^{J_\mu}(y)-\dot{d}_{\mu\nu}^{J_\mu}(z)\,d_{\mu\nu}^{J_\mu+1}(y)](z-y)$$
$$-d_{\mu\nu}^{J_\mu+1}(z)\,d_{\mu\nu}^{J_\mu}(y)+d_{\mu\nu}^{J_\mu}(z)\,d_{\mu\nu}^{J_\mu+1}(y)\}$$
$$\times\{(z-y)[d_{\mu\nu}^{J_\mu+1}(z)\,d_{\mu\nu}^{J_\mu}(y)-d_{\mu\nu}^{J_\mu}(z)\,d_{\mu\nu}^{J_\mu+1}(y)]\}^{-1} \tag{31}$$

Proof. The proof of these results is similar to that of Theorem 2 and will be omitted. ■

Corollary 5. Let $f^{[\mu_0]}$ be the helicity amplitude of channel $[\mu_0]$, and let $f^{[\mu]}\in H^{[\mu_0]}$, where $H^{[\mu_0]}$ is an RKHS with the RK (26a), (26b).

(i) If $\sigma_{\text{el}}^{[\mu_0]}$ and $\sigma_T^{[\mu_0]}$ are given, then any cutoff J on the total angular momentum j must obey the inequality

$$(J+1)^2\geq[(\sigma_T^{[\mu_0]})^2/4\pi\lambda^2\sigma_{\text{el}}^{[\mu_0]}]+\mu^2 \tag{32a}$$

(ii) The equality holds in (32a) if and only if $f^{[\mu_0]}$ is the optimal amplitude

$$f^{[\mu_0]}(x)=i\frac{\sigma_T^{[\mu_0]}}{4\pi\lambda}\frac{K^{[\mu_0]}(x,1)}{K^{[\mu_0]}(1,1)}$$
$$=i\frac{\sigma_T^{[\mu_0]}}{4\pi\lambda}\frac{1}{J+1}\frac{d_{\mu\mu}^{J+1}(x)-d_{\mu\mu}^{J}(x)}{x-1} \tag{32b}$$

where

$$J = \text{integer (or half-integer)}\{[((\sigma_T^{[\mu_0]})^2/4\pi\lambda\!\!\!^{-2}\sigma_{\text{el}}^{[\mu_0]}) + \mu^2]^{1/2} - 1\} \qquad (32c)$$

(iii) The forward diffraction peak of the process (1) described by the optimal amplitude (32b), (32c) possesses the scaling property

$$\frac{1}{(d\sigma^{[\mu_0]}/d\Omega)(1)} \frac{d\sigma^{[\mu_0]}}{d\Omega}(x) = \left[\frac{J_1(\tau_t)}{\tau_t}\right]^2 \quad \text{for small} \quad \tau_t \qquad (33a)$$

where

$$\frac{d\sigma^{[\mu_0]}}{d\Omega}(1) = \left(\frac{\sigma_T^{[\mu_0]}}{4\pi\lambda\!\!\!^{-}}\right)^2 \qquad (33b)$$

and

$$\tau_t = 2(|t|\, b_t^{[\mu_0]})^{1/2} = \left\{|t|\left[\frac{(\sigma_T^{[\mu_0]})^2}{4\pi\sigma_{\text{el}}^{[\mu_0]}} - \lambda\!\!\!^{-2}\right]\right\}^{1/2} \qquad (33c)$$

(iv) If $\mu_a = \mu_b = \mu'_a = \mu'_b$, the backward diffraction peak of the process (1) described by the optimal state (32b), (32c) possesses the scaling property

$$\frac{1}{(d\sigma^{[\mu_0]}/d\Omega)(-1)} \frac{d\sigma^{[\mu_0]}}{d\Omega}(x) = \left[\frac{J_1(\tau_u)}{\tau_u}\right]^2 \quad \text{for small} \quad \tau_u \qquad (33d)$$

where

$$\frac{d\sigma^{[\mu_0]}}{d\Omega}(-1) = \frac{\sigma_{\text{el}}^{[\mu_0]}}{4\pi} \qquad (33e)$$

and

$$\tau_u = 2[|u|\, b_u^{[\mu_0]}]^{1/2} = \left\{2|u|\left[\frac{(\sigma_T^{[\mu_0]})^2}{4\pi\sigma_{\text{el}}^{[\mu_0]}} - \lambda\!\!\!^{-2}\right]\right\}^{1/2} \qquad (33f)$$

Proof. The results (i)–(ii) are obtained from (28)–(30) and the Wick-like inequality

$$\left(\frac{\sigma_T^{[\mu_0]}}{4\pi\lambda}\right)^2 \leq \frac{d\sigma^{[\mu_0]}}{d\Omega}(1)$$

since from (26b)

$$K^{[\mu_0]}(1, 1) = \tfrac{1}{2}[(J_\mu + 1)^2 - \mu^2]$$

The optimal predictions (33a)–(33c) and (33d)–(33f) are obtained in a straightforward way using the following properties of the rotation functions:

$$d^j_{\mu\nu}(\pi-\theta)=(-1)^{j-\mu}\, d^j_{-\mu\nu}(\theta)=(-1)^{j-\mu}\, d^j_{\nu-\mu}(\theta) \tag{34a}$$

$$d^j_{\mu\nu}(x)=(-1)^{\mu-\nu}\, d^j_{-\mu-\nu}(x)=(-1)^{\mu-\nu}\, d^j_{\nu\mu}(x) \tag{34b}$$

$$d^j_{\mu\nu}(x)=(-1)^{\Lambda}\left[\frac{(j+M)!(j-N)!}{(j+N)!(j-M)!}\right]^{1/2} \times\left(\frac{1-x}{2}\right)^{|\mu-\nu|}\left(\frac{1+x}{2}\right)^{|\mu-\nu|} P^{(|\mu-\nu|,|\mu+\nu|)}_{j-M}(x) \tag{34c}$$

$$d^j_{\mu\nu}(x)\approx J_{|\mu-\nu|}\left[(2j+1)\sin\frac{\theta}{2}\right] \qquad \text{for small } \theta \text{ and } j\gg 1 \tag{34d}$$

where $P^{(\alpha,\beta)}_{j-M}$ are the Jacobi polynomials, and

$$M\equiv\max[|\mu|,|\nu|],\quad N\equiv\min[|\mu|,|\nu|],\quad \Lambda\equiv\tfrac{1}{2}(\mu-\nu-|\mu-\nu|). \tag{34e}$$

and $J_{|\mu-\nu|}(x)$ are the Bessel functions of order $|\mu-\nu|$. ■

Corollary 6. Let $f^{[\mu]}$ be an element of the RKHS $H^{[\mu]}$ that has the RK (26a), (26b), where $[\mu]$ here denotes only those channels for which $\mu=-\nu$.

(ii) If $\sigma^{[\mu]}_{\text{el}}$ and $(d\sigma^{[\mu]}/d\Omega)(-1)$ are given, then any cutoff J_μ on the total angular momentum must obey the bound

$$(J_\mu+1)^2\geq\frac{4\pi}{\sigma^{[\mu]}_{\text{el}}}\frac{d\sigma^{[\mu]}}{d\Omega}(-1)+\mu^2 \tag{35a}$$

(ii) The equality holds in (35a) if and only if $f^{[\mu]}$ is the optimal amplitude

$$f^{[\mu]}(x)=f^{[\mu]}(-1)\frac{K^{[\mu]}(x,-1)}{K^{[\mu]}(-1,-1)} \tag{35b}$$

where

$$J_\mu=\text{integer (or half-integer)}\left\{\left[\frac{4\pi}{\sigma^{[\mu]}_{\text{el}}}\frac{d\sigma^{[\mu]}}{d\Omega}(-1)+\mu^2\right]^{1/2}-1\right\} \tag{35c}$$

(iii) The backward diffraction peak of the process (1) described by the optimal amplitude (35b), (35c) possesses the scaling property

$$\frac{1}{(d\sigma^{[\mu]}/d\Omega)(-1)}\frac{d\sigma^{[\mu]}}{d\Omega}(x)=\left[\frac{J_1(\tau_u)}{\tau_u}\right]^2 \qquad \text{for small}\quad \tau_u \tag{36a}$$

where

$$\tau_u=2(|u|\,b^{[\mu]}_u)^{1/2}=\left\{\lambda^2|u|\left[\frac{4\pi}{\sigma^{[\mu]}_{\text{el}}}\frac{d\sigma^{[\mu]}}{d\Omega}(-1)-1\right]\right\}^{1/2} \tag{36b}$$

(iv) If $\mu_a = \mu_b$ and $\mu'_a = \mu'_b$, then the forward diffraction peak of the process (1) described by the optimal amplitude (35b), (35c) possesses the scaling property

$$\frac{1}{(d\sigma^{[\mu]}/d\Omega)(1)} \frac{d\sigma^{[\mu]}}{d\Omega}(x) = \left[\frac{J_1(\tau_t)}{\tau_t}\right]^2 \qquad \text{for small } \tau_t \tag{37a}$$

where

$$\frac{d\sigma^{[\mu]}}{d\Omega}(1) = \frac{\sigma_{\text{el}}^{[\mu]}}{4\pi} \tag{37b}$$

and

$$\tau_t = 2(|t|\, b_t^{[\mu]})^{1/2} = \left\{2\lambda\!\!\!^-{}^2|t|\left[\frac{4\pi}{\sigma_{\text{el}}^{[\mu]}} \frac{d\sigma^{[\mu]}}{d\Omega}(-1) - 1\right]\right\}^{1/2} \tag{37c}$$

Proof. The results (35a), (35c) are obtained from equations (28)-(30) for $y = -1$, since $K^{[\mu]}(-1, -1) = \frac{1}{2}[(J_\mu + 1)^2 - \mu^2]$ for all the channels $[\mu]$ for which $\mu = -\nu$. The predictions (36a), (36b) and (37a), (37b) are obtained using equations (35b), (35c), and (34d). ■

Theorem 5. Assume that each helicity amplitude $f^{[\mu]}$ of the process (1) is an element of the Hilbert space $H^{[\mu]}$ that possesses the reproducing kernel (26a), (26b).

(i) Then, if σ_{el} and σ_T [see equations (24b), (24c)] are given, any cutoff J on the total angular momentum must obey the inequality

$$(J+1)^2 \geq \frac{\sigma_T^2}{4\pi\lambda\!\!\!^-{}^2\sigma_{\text{el}}} + \langle\mu^2\rangle \tag{38a}$$

where

$$\langle\mu^2\rangle = \frac{1}{(2s_a+1)(2s_b+1)} \sum_{[\mu_0]} \mu^2 \tag{38b}$$

(ii) The equality holds in (43a), (43b) if and only if

$$f^{[\mu]}(x) = 0 \qquad \text{for all } [\mu] \neq [\mu_0] \tag{39a}$$

$$\begin{aligned} f^{[\mu_0]}(x) &= i\frac{\sigma_T^{[\mu_0]}}{4\pi\lambda\!\!\!^-} \frac{K^{[\mu_0]}(x,1)}{K^{[\mu_0]}(1,1)} \\ &= i\frac{\sigma_T^{[\mu_0]}}{4\pi\lambda\!\!\!^-} \frac{1}{J+1} \frac{d_{\mu\mu}^{J+1}(x) - d_{\mu\mu}^{J}(x)}{x-1} \end{aligned} \tag{39b}$$

for all helicity-conserving channels $[\mu_0]$, where

$$J = \text{integer (or half-integer)}\left\{\left(\frac{\sigma_T^2}{4\pi\lambda\!\!\!^-{}^2\sigma_{\text{el}}} + \langle\mu^2\rangle\right)^{1/2} - 1\right\} \tag{39c}$$

and

$$\sigma_T^{[\mu_0]} = \sigma_T + \frac{4\pi\lambda\!\!\!^{-2}\sigma_{el}}{\sigma_T}(\langle\mu^2\rangle - \mu^2) \tag{39d}$$

(iii) The forward diffraction peak of the process (1) described by the optimal amplitudes (39a), (39d) possesses the scaling property

$$\frac{1}{(d\sigma/d\Omega)(1)}\frac{d\sigma}{d\Omega}(x) = \left[\frac{J_1(\tau_t)}{\tau_t}\right]^2 \qquad \text{for small} \quad \tau_t \tag{40a}$$

where

$$\frac{d\sigma}{d\Omega}(1) = \left(\frac{\sigma_T}{4\pi\lambda\!\!\!^{-}}\right)^2 + \lambda\!\!\!^{-2}\frac{\sigma_{el}^2}{\sigma_T^2}(\langle\mu^4\rangle - \langle\mu^2\rangle^2) \tag{40b}$$

$$\tau_t = 2(|t|b_t)^{1/2} = \left[|t|\left(\frac{\sigma_T S}{4\pi\sigma_{el}} - \lambda\!\!\!^{-2}\right)\right]^{1/2} \tag{40c}$$

$$S \equiv \frac{1}{(2s_a+1)(2s_b+1)} \frac{\left\{\sum_{[\mu_0]} \sigma_T^{[\mu_0]}\left(\frac{\sigma_T^{[\mu_0]}}{4\pi\lambda\!\!\!^{-}}\right)^2\right\}}{\left[\sigma_T \frac{d\sigma}{d\Omega}(1)\right]} \tag{40d}$$

(iv) Let $n_0 \neq 0$ be the number of channels with $\mu = -\nu = 0$. Then the backward diffraction peak of the process (1) described by the optimal amplitudes (39a)-(39d) possesses the scaling property

$$\frac{1}{(d\sigma/d\Omega)(-1)}\frac{d\sigma}{d\Omega}(x) = \left[\frac{J_1(\tau_u)}{\tau_u}\right]^2 \qquad \text{for small} \quad \tau_u \tag{41a}$$

where

$$\frac{d\sigma}{d\Omega}(-1) = \frac{n_0}{(2s_a+1)(2s_b+1)}\frac{\sigma_{el}}{4\pi}\left(1 + \frac{4\pi\lambda\!\!\!^{-2}\sigma_{el}}{\sigma_T^2}\langle\mu^2\rangle\right) \tag{41b}$$

$$\tau_u = 2(|u|b_u)^{1/2} = \left\{2|u|\lambda\!\!\!^{-2}\left(\frac{\sigma_T^2}{4\pi\lambda\!\!\!^{-2}\sigma_{el}} + \langle\mu^2\rangle - 1\right)\right\}^{1/2} \tag{41c}$$

Proof. The results (38a), (38b), and (39a)-(39c) are obtained by using the Lagrange multiplier method (Wilde and Beightler, 1967; Einhorn and Blankenbecler, 1971) to solve the problem (I): *minimize* σ_{el} *subject to* $f^{[\mu]} \in H^{[\mu]}$ *for all channels* $[\mu]$ *when* σ_T *is given* (see Ion, 1982b). We introduce the variational function

$$L = \sum_{[\mu]}\sum_{|\mu|}^{J}(2j+1)[(a_j^{[\mu]})^2 + (r_j^{[\mu]})^2]$$
$$+ \alpha\left\{(2s_a+1)(2s_b+1)\frac{\sigma_T}{4\pi} - \lambda\!\!\!^{-}\sum_{[\mu_0]}\sum_{|\mu|}^{J}(2j+1)a_j^{[\mu_0]}\right\}$$

where the $a_j^{[\mu]}$ and $r_j^{[\mu]}$ denote the real and imaginary parts of the partial helicity amplitudes $f_j^{[\mu]}$. Thus, using the variational equations, we obtain the results (38a), (38b), and (39a)-(39c), since the stationary solution $(\overset{*}{r}{}_j^{[\mu]}, \overset{*}{a}{}_j^{[\mu]}, \overset{*}{\alpha})$ of the problem (I) is given by

$$\overset{*}{r}{}_j^{[\mu]}=0, \qquad \overset{*}{a}{}_j^{[\mu]}=\lambda\overset{*}{\alpha}\delta_{[\mu][\mu_0]} \qquad \text{for all} \quad |\mu|\leq j\leq J$$

$$\overset{*}{r}{}_j^{[\mu]}=\overset{*}{a}{}_j^{[\mu]}=0 \qquad \text{for all} \quad j\geq J+1$$

where

$$\overset{*}{\alpha}=\frac{\sigma_{\text{el}}}{\sigma_T}=\frac{\sigma_{\text{el}}^{[\mu_0]}}{\sigma_T^{[\mu_0]}}$$

and

$$(J+1)^2-\mu^2=\frac{\sigma_T}{4\pi\lambda^2\sigma_{\text{el}}}\sigma_T^{[\mu_0]}=\frac{(\sigma_T^{[\mu_0]})^2}{4\pi\lambda^2\sigma_{\text{el}}^{[\mu_0]}}$$

Then, the results (iii) and (iv) are derived using equations (20a)-(20c), (34a)-(34e), (39a)-(39d), and

$$\begin{aligned} b_t^{[\mu_0]} &\equiv \frac{d}{dt}\left[\ln\frac{d\sigma^{[\mu_0]}}{d\Omega}(x)\right]\bigg|_{t=0}=\frac{\lambda^2}{4}[(J+1)^2-\mu^2-1] \\ &=\frac{\lambda^2}{4}\left(\frac{\sigma_T^2}{4\pi\lambda^2\sigma_{\text{el}}}+\langle\mu^2\rangle-\mu^2-1\right) \end{aligned}$$

and

$$\begin{aligned} b_u^{[\mu_0]} &\equiv \frac{d}{du}\left[\ln\frac{d\sigma^{[\mu_0]}}{d\Omega}(x)\right]\bigg|_{u=0}=\frac{\lambda^2}{2}(J+1)^2 \\ &=\frac{\lambda^2}{2}\left(\frac{\sigma_T}{4\pi\lambda^2\sigma_{\text{el}}}+\langle\mu^2\rangle-1\right) \end{aligned}$$

only for $\mu=-\nu=0$, since $(d\sigma^{[\mu_0]}/d\Omega)(-1)=0$ for all channels with $m=\nu\neq 0$. Also, we have used the results

$$\begin{aligned} \frac{d}{dx}\left[\frac{K^{[\mu_0]}(x,1)}{K^{[\mu_0]}(1,1)}\right]\bigg|_{z=1} &=\frac{1}{2(J+1)}[\ddot{d}{}_{\mu\mu}^{J+1}(1)-\ddot{d}{}_{\mu\mu}^{J}(1)] \\ &=\tfrac{1}{4}([J+1)^2-\mu^2-1] \end{aligned}$$

since

$$\begin{aligned} \ddot{d}{}_{\mu\mu}^{J}(1) &=\frac{d^2}{dx^2}[d_{\mu\mu}^J(x)]|_{x=1} \\ &=\frac{\mu(\mu-1)}{2}+\frac{\mu}{2}[J(J+1)-\mu(\mu+1)] \\ &\quad+\tfrac{1}{8}[J(J+1)-(\mu+1)(\mu+2)][J(J+1)-\mu(\mu+1)] \quad \blacksquare \end{aligned}$$

Remark 1. The model-independent results (12a) and (32a) were obtained by Rarita and Schwed (1958) and Ion (1982b), respectively.

Remark 2. The results (40a), (40b) include in a more general and exact form the scaling variables $|t|\sigma_T^2/\sigma_{el}$ and $|t|\sigma_T$ introduced by Singh and Roy (1970), Cornille and Martin (1976), Dias de Deus (1973), and Buras and Dias de Deus (1974), since at high energies [see equations (39d), (40a)-(40d)]

$$\sigma_T^{[\mu_0]} \to \sigma_T, \qquad \frac{d\sigma}{d\Omega}(1) \to \left(\frac{\sigma_T}{4\pi\lambda\!\!\!^{-}}\right)^2 \qquad S \to 1$$

Remark 3. The scattering phenomena described by the optimal helicity amplitudes (39a)-(39d) are *s-channel helicity-conserving phenomena.* Their angular distributions have diffractive patterns very sensitive to the values of the optimal curoff parameter J and at high energies ($J \gg 1$) possess the scaling properties (40a)-(40d) and (41a)-(41c). The idea that s-channel helicity conservation is a universal phenomenon associated with diffractive processes was proposed by Gilman (1970) as a speculation stimulated by the finding that s-channel helicity is conserved in the reaction $\gamma p \to \rho^0 p$ (Ballam *et al.*, 1970). Harari and Zarmi (1970) and Bialas *et al.* (1970) presented experimental evidence in support of this idea in πN scattering and in reactions involving isobar production.

Corollary 7. Assume that each helicity amplitude $f^{[\mu]}$ is an element of the RKHS $H^{[\mu]}$ that possesses the RK $K^{[\mu]}$, (26a), (26b).

(i) If σ_{el} and $(d\sigma/d\Omega)(-1)$ are given, then any cutoff J_μ on the total angular momentum must obey the bound

$$(J_\mu + 1)^2 \geq \frac{4\pi}{\sigma_{el}} \frac{d\sigma}{d\Omega}(-1) + \mu^2 \tag{42}$$

(ii) The equality holds in (42) if and only if $f^{[\mu]}(x)$ is the optimal amplitude

$$\begin{aligned} f^{[\mu]}(x) &= f^{[\mu]}(-1)\frac{K^{[\mu]}(x, 1)}{K^{[\mu]}(-1, -1)} \\ &= f^{[\mu]}(-1)\frac{1}{J_\mu + 1} \\ &\quad \times \frac{d^{J_\mu+1}_{\mu-\mu}(x)\, d^{J_\mu}_{\mu-\mu}(-1) - d^{J_\mu}_{\mu-\mu}(x)\, d^{J_\mu+1}_{\mu-\mu}(-1)}{x+1} \\ &= f^{[\mu]}(-1)\frac{1}{J_\mu + 1}\frac{d^{J_\mu+1}_{\mu-\mu}(x) - d^{J_\mu}_{\mu-\mu}(x)}{x+1} \end{aligned} \tag{43a}$$

where

$$J_\mu = \text{integer (or half-integer)}\left\{\left[\frac{4\pi}{\sigma_{\text{el}}}\frac{d\sigma}{d\Omega}(-1)+\mu^2\right]^{1/2}-1\right\} \tag{43b}$$

for all channels $[\mu]$ for which $\mu=-\nu$ [see equation (34a)].

(iii) The backward diffraction peak of the angular distribution of the process (1) described by the helicity amplitudes (43a), (43b) posesses the scaling property

$$\frac{1}{(d\sigma/d\Omega)(-1)}\frac{d\sigma}{d\Omega}(x)=\left[\frac{J_1(\tau_u)}{\tau_u}\right] \quad \text{for small} \quad \tau_u \tag{44a}$$

where

$$\tau_u = 2(|u|b_u)^{1/2}\left\{\lambda\!\!\!^{-2}|u|\left[\frac{4\pi}{\sigma_{\text{el}}}\frac{d\sigma}{d\Omega}(-1)-1\right]\right\}^{1/2} \tag{44b}$$

(iv) If $n_0\neq 0$ is the number of channels $[\mu]$ with $\mu=\nu=0$, then the forward peak of the angular distribution of the process (1) described by the optimal scattering amplitudes (43a), (43b) possesses the scaling property

$$\frac{1}{(d\sigma/d\Omega)(1)}\frac{d\sigma}{d\Omega}(x)=\left[\frac{J_1(\tau_t)}{\tau_t}\right]^2 \quad \text{for small} \quad \tau_t \tag{45a}$$

where

$$\frac{d\sigma}{d\Omega}(1)=\frac{n_0}{(2s_a+1)(2s_b+1)}\frac{\sigma_{\text{el}}}{4\pi} \tag{45b}$$

and

$$\tau_t = 2(|t|b_t)^{1/2}=\left\{2\lambda\!\!\!^{-2}|t|\left[\frac{4\pi}{\sigma_{\text{el}}}\frac{d\sigma}{d\Omega}(-1)-1\right]\right\}^{1/2} \tag{45c}$$

Proof. The results (i) and (ii) are obtained from Theorem 2 of Ion (1985) for $y=1$ when the RK is given by (26a), (26b). An important step here is the equality

$$\frac{4\pi}{\sigma_{\text{el}}}\frac{d\sigma}{\delta\Omega}(-1)=\frac{4\pi}{\sigma_{\text{el}}^{[\mu]}}\frac{d\sigma^{[\mu]}}{d\Omega}(-1)=(J_\mu+1)^2-\mu^2=2K^{[\mu]}(-1,-1) \tag{45'}$$

for all channels $[\mu]$ with $\mu=-\nu$ for which $K^{[\mu]}(-1,-1)\neq 0$. Then, the results (44a), (44b), and (45a)-(45c) are derived using the definitions (20a), equation (45′), the property (34d), and the results

$$b_u^{[\mu]} = \frac{d}{dt}\left[\ln \frac{d\sigma^{[\mu]}}{d\Omega}(x)\right]\bigg|_{u=0} = \frac{\lambda\!\!\!^{-2}}{4}\left[\frac{4\pi}{\sigma_{\rm el}}\frac{d\sigma}{d\Omega}(-1) - 1\right]$$

for any channel $[\mu]$ with $\mu = -\nu$, and

$$b_t^{[\mu]} = \frac{d}{dt}\left[\ln \frac{d\sigma^{[\mu]}}{d\Omega}(x)\right]\bigg|_{t=0} = \frac{\lambda\!\!\!^{-2}}{2}\left[\frac{4\pi}{\sigma_{\rm wl}}\frac{d\sigma}{d\Omega}(-1) - 1\right]$$

for each channel $[\mu]$ with $\mu = -\nu = 0$ ■

Corollary 8. Assume that each helicity amplitude $f^{[\mu]}$ is an element of the RKHS $H^{[\mu]}$ with the RK $K^{[\mu]}$, (26a), (26b).

(i) If $\sigma_{\rm el}$ and $(d\sigma/d\Omega)(1)$ are given, then any cutoff J_μ on the total angular momentum must obey the bound

$$(J_\mu + 1)^2 \geq \frac{4\pi}{\sigma_{\rm el}}\frac{d\sigma}{d\Omega}(1) + \mu^2 \tag{46}$$

(ii) The equality holds in (46) if and only if $f^{[\mu]}(x)$ is the optimal amplitude

$$f^{[\mu]}(x) = f^{[\mu]}(1)\frac{K^{[\mu]}(x,1)}{K^{[\mu]}(1,1))} = f^{[\mu]}(1)\frac{1}{J_\mu + 1}\frac{d_{\mu\mu}^{J_\mu+1}(x) - d_{\mu\mu}^{J_\mu}(x)}{x-1} \tag{47a}$$

where

$$J_\mu = \text{integer (or half-integer)}\left\{\left[\frac{4\pi}{\sigma_{\rm el}}\frac{d\sigma}{d\Omega}(1) + \mu^2\right]^{1/2} - 1\right\} \tag{47b}$$

(iii) The forward diffraction peak of the optimal state (47a), (47b) possesses the scaling property

$$\frac{1}{(d\sigma/d\Omega)(1)}\frac{d\sigma}{d\Omega}(x) = \left[\frac{J_1(\tau_t)}{\tau_t}\right]^2 \qquad \text{for small} \quad \tau_t \tag{48a}$$

where

$$\tau_t = 2(|t|b_t)^{1/2} = \left\{\lambda\!\!\!^{-2}|t|\left[\frac{4\pi}{\sigma_{\rm el}}\frac{d\sigma}{d\Omega}(1) - 1\right]\right\}^{1/2} \tag{48b}$$

(iv) Let $n_0 \neq 0$ be the number of channels $[\mu]$ with $\mu = \nu = 0$. Then the backward peak of the angular distribution of the process (1) described by the optimal state (47a), (47b) possesses the scaling property

$$\frac{1}{(d\sigma/d\Omega)(-1)}\frac{d\sigma}{d\Omega}(x) = \left[\frac{J_1(\tau_u)}{\tau_u}\right]^2 \qquad \text{for small} \quad \tau_u \tag{49a}$$

where

$$\frac{d\sigma}{d\Omega}(-1) = \frac{n_0}{(2s_a+1)(2s_b+1)}\frac{\sigma_{el}}{4\pi} \tag{49b}$$

and

$$\tau_u = 2(|t|b_u)^{1/2} = \left\{2\lambda\!\!\!^{-2}|t|\left[\frac{4\pi}{\sigma_{el}}\frac{d\sigma}{d\Omega}(1)-1\right]\right\}^{1/2} \tag{49c}$$

Proof. The results (i)-(iii) are obtained in Theorem 3 of Ion (1985), while the predictions (iv) can be obtained in a straightforward way using the definition (20a), the property (34d), and equations (47a), (47b). ■

Remark 4. Parida (1979) has pointed out that the scaling of the experimental data in the variable $|t|b_t$ is much better than the scaling in the variables $|t|\sigma_T$ and $||t|\sigma_T^2/\sigma_{el}$. Figure 1 and Tables 1a and 1b of Ion (1982a) present the experimental data on the logarithmic slopes of the forward diffraction peak for the most usual reactions [e.g., $pp \to pp$; $\bar{p}p \to \bar{p}p$; $K^{\pm}p \to K^{\pm}p$; $\pi^{\pm}p \to \pi^{\pm}p$] in comparison with the optimal prediction

$$b_t = \frac{\lambda\!\!\!^{-2}}{4}\left[\frac{4\pi}{\sigma_{el}}\frac{d\sigma}{d\Omega}(1)-1\right] \tag{50}$$

of the optimal state (47a), (47b). The prediction (50) is satisfied experimentally to a surprising accuracy for all pp, $\bar{p}p$, $K^{\pm}p$, $\pi^{\pm}p$ scattering processes at all energies higher than 2 GeV. Moreover, from Table 2 of Ion (1982a) one can see that the experimental slopes for $\bar{p}p$ scattering are in agreement with the optimal result (50) even in the low-energy region.

4. CONCLUSIONS

The present paper is a continuation and a extension of our previous results (Ion and Scutaru, 1985; Ion, 1985) in which the two-body scattering amplitude is assumed to be an element of a functional Hilbert space defined on $[-1, +1]$ called the RKHS. We have shown that the RKHSs have many special properties, such as (a) autoreproducibility of RK (b) uniqueness of RK, (c) uniqueness of RKHS, (d) completeness of the set $\{K_y^{[\mu]}, y \in [-1, +1]\}$, (e) pointwise convergence in RKHSs, (f) positiveness of RK, (g) minimum norm of RK, (h) smoothness, and (i) overcompleteness of the full set $\{K_y^{[\mu]}, y \in [-1, +1]\}$. Therefore, defining the optimal state by equation (7) [or (23)], we obtain that a number of essential characteristic features common to all the optimal states are direct consequence of the above RK properties. For any spacific example of optimal states, a corresponding set of additional properties holds true.

Our conclusions may be summarized as follows:

(i) If the scattering amplitude f of the process (1) is an element of the finite-dimensional subspace $L^2(-1, +1)$, then f can be developed in terms of a finite number of partial amplitudes (9) and the RKHS H defined on $[-1, +1]$ possesses the RK (10a), (10b). Then, any cutoff parameter L on the total angular momentum allowed in the process (1) must obey the bounds (11a), (12a), (15a), and (32) (Ion and Scutaru, 1985). These bounds are saturated if and only if the scattering amplitudes are the optimal states (11b), (11c), (12b), (12c), (15b), (15c), and (33a), (33b) (Ion and Scutaru, 1985), respectively. The scattering phenomena described by these optimal states have diffractive patterns very sensitive to the values of the cutoff parameter L (e.g., the number of maxima of the differential cross section in the entire $\cos\theta$ interval is $L+1$ and which at high energies $(L \gg 1)$ possesses the scaling properties (13a)-(13c), (14a)-(14c), (16a)-(16c), (17a), (17b), and (35a), (35b) (Ion and Scutaru, 1985). All these results are extended to the scattering of particles with arbitrary spins in Section 3.

(ii) If each helicity amplitude $f^{[\mu]}$ of the process (1) is an element of the RKHS $H^{[\mu]}$ with the RK (26a), (26b), then any cutoff on the total angular momentum must obey the bounds (28), (32a), (35a), (38a), (38b), (42), and (46). The equality holds in each of the above inequalities if and only if $f^{[\mu]}$ is the optimal state (29), (32b), (32c), (35b), (35c), (39a)-(39c), (43a), (43b), (47a), (47b), respectively. The angular distributions of these optimal states possess remarkable scaling properties [(33a)-(33c), (33d)-(33f), (36a), (36b), (37a), (37b), (40a)-(40d), (41a)-(41c), (44a), (44b), (45a)-(45c), (48a), (48b), (49a)-(49c)]. The results (40a), (40b) include in a more general and exact form the scaling variables $|t|\sigma_T^2/4\pi\sigma_{el}$ and $||t|\sigma_T$ introduced by many authors (see Remark 2). The scattering phenomena described by the optimal amplitudes (39a)-(39d) are s-channel helicity-conserving phenomena (see Remark 3).

(iii) The optimal state dominance as well as the above scaling and s-channel helicity conservation laws are experimentally well established (see Ion, 1982; Parida 1979) for all pp, $\bar{p}p$, $\pi^{\pm}p$, $K^{\pm}p$ scattering at all energies higher than 2 GeV.

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